G-CHARACTER VARIETIES FOR $G = \mathrm{SO}(n, \mathbb{C})$ AND OTHER NOT SIMPLY CONNECTED GROUPS

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Abstract. We describe the relation between G-character varieties, $X_G(\Gamma)$, and G/H-character varieties, where H is a finite, central subgroup of G. In particular, we find finite generating sets of $\mathbb{C}[X_{G/H}(\Gamma)]$ for classical groups G and H as above. Using this approach we find an explicit description of $\mathbb{C}[X_{SO(4,\mathbb{C})}(F_2)]$ for the free group on two generators, F_2.

In the second part of the paper, we prove several properties of $SO(2n, \mathbb{C})$-character varieties. This is a particularly interesting class of character varieties because unlike for all other classical groups G, the coordinate rings $\mathbb{C}[X_G(\Gamma)]$ are generally not generated by τ_γ, for $\gamma \in \Gamma$, for $G = SO(2n, \mathbb{C})$. We prove that for every finite group Γ, the coordinate ring $\mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)]$ is generated by $\tau_{\gamma,V}$ for all $\gamma \in \Gamma$ and all representations V of $SO(2n, \mathbb{C})$. We also prove that this is not the case for $n = 2$ and groups Γ of corank ≥ 2.

1. Introduction

Let G will be an affine reductive algebraic group over $\mathbb{C}$.[1] For every group Γ generated by some $\gamma_1, ..., \gamma_N$, the space of all G-representations of Γ forms an algebraic subset, $Hom(\Gamma, G)$, of G^N on which G acts by conjugating representations. The categorical quotient of that action

$$X_G(\Gamma) = Hom(\Gamma, G)//G$$

is the G-character variety of Γ, cf. [LM, S1] and the references within.

In this paper, we study character varieties for not simply connected groups. In Sections 2-3 we describe the relations between $X_G(\Gamma)$ and $X_{G/H}(\Gamma)$, for all finite, central subgroups H of G. In particular, we find finite generating sets of $\mathbb{C}[X_{G/H}(\Gamma)]$ for classical groups G and H as above, cf. Theorem 3, Corollary 4, and Proposition 5. Using this approach we find an explicit description of $\mathbb{C}[X_{SO(4,\mathbb{C})}(F_2)]$ for the free group on two generators, F_2, cf. Proposition 18.

In Sections 4-7, we use the above results to prove several properties of the $SO(2n, \mathbb{C})$-character varieties. This is a particularly interesting class of character varieties because unlike for all other classical groups G, the

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[1]Throughout the paper, the field of complex numbers can be replaced an arbitrary algebraically closed field of characteristic zero or of characteristic large enough, depending on n below.

coordinate ring $\mathbb{C}[X_G(\Gamma)]$ does not always coincide with the G-trace algebra of Γ, $\mathcal{T}_G(\Gamma)$, for $G = SO(2n,\mathbb{C})$, cf. [S2].

Let us recall the definition of trace algebras. For a representation V of G and $\gamma \in \Gamma$ let

$$\tau_{\gamma,V}([\rho]) = tr(\phi_V\rho(\gamma)),$$

where ϕ_V is the homomorphism $\Gamma \to GL(V)$ induced by V. Then $\tau_{\gamma,V}$ is a regular function on $X_G(\Gamma)$. For a matrix group $G \subset GL(n,\mathbb{C})$, the G-trace algebra, $\mathcal{T}_G(\Gamma)$, is the $\mathbb{C}$-subalgebra of $\mathbb{C}[X_G(\Gamma)]$ generated by $\tau_\gamma = \tau_{\gamma,\mathbb{C}^n}$, for all $\gamma \in \Gamma$.

Let us also recall from [S2], that the full G-trace algebra, $\mathcal{FT}_G(\Gamma)$, is the $\mathbb{C}$-subalgebra of $\mathbb{C}[X_G(\Gamma)]$ generated by $\tau_{\gamma,V}$, for all representations V of G and for all $\gamma \in \Gamma$. Addressing a question posted in [S2], we investigate the extensions

$$\mathcal{T}_{SO(2n,\mathbb{C})}(\Gamma) \subset \mathcal{FT}_{SO(2n,\mathbb{C})}(\Gamma) \subset \mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)].$$

By [S2, Thm 8][2], $\mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)]$ is a $\mathcal{T}_{SO(2n,\mathbb{C})}(\Gamma)$-algebra generated by $Q_{2n}(g_1,...,g_n)$ for all words $g_1,...,g_n$ in $\gamma_1,...,\gamma_N$ of length at most $\nu_n - 1$ in which the number of inverses is not larger than the half of the length of the word. (See [S2] for the definition of ν_n.) Here is a stronger version of that result:

Proposition. *(See Corollary 10.)* $\mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)]$ *is a* $\mathcal{T}_{SO(2n,\mathbb{C})}(\Gamma)$*-module generated by* 1 *and by* $Q_{2n}(g_1,...,g_n)$ *for words* $g_1,...,g_n \in \Gamma$ *in* $\gamma_1,...,\gamma_N$ *of length at most* $\nu_n - 1$ *in which the number of inverses is not larger than the half of the length of the word.*

Furthermore, for finite groups Γ we have:

Theorem. *(See Theorem 12.) Let* Γ *be finite. If* $Q_n(g_1,...,g_n) \neq 0$ *for some* $g_1,...,g_n \in \Gamma$ *then* $\mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)]$ *is generated by* 1 *and by* $Q_n(g_1,...,g_n)$ *as a* $\mathcal{T}_{SO(2n,\mathbb{C})}(\Gamma)$*-module.*

Consequently, $\mathcal{FT}_{SO(2n,\mathbb{C})}(\Gamma) = \mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)]$ for every finite group Γ.

The determination of whether the extension $\mathcal{FT}_{SO(2n,\mathbb{C})}(\Gamma) \subset \mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)]$ is proper for infinite Γ is a delicate problem. In particular, by the result of Vinberg, [Vi], $\mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)]$ is the integral closure of $\mathcal{FT}_{SO(2n,\mathbb{C})}(\Gamma)$ in its field of fractions for Γ free.

We analyze this extension in detail for $\Gamma = F_2$ and $n = 2$ in Sections 4-7. In particular we prove:

Theorem. *(See Theorem 14.)*
(1) $\mathbb{C}[X_{SO(4,\mathbb{C})}(F_2)]$ *is a* $\mathcal{T}_{SO(4,\mathbb{C})}(F_2)$*-module generated by* $1, Q_4(\gamma_i,\gamma_j)$*, for* $1 \le i \le j \le 2$*, and by* $Q_4(\gamma_1\gamma_2,\gamma_1\gamma_2)$, $Q_4(\gamma_1\gamma_2^{-1},\gamma_1\gamma_2^{-1})$, $Q_4(\gamma_1\gamma_2^{-1},\gamma_2)$, $Q_4(\gamma_2\gamma_1^{-1},\gamma_1)$.
(2) $\mathbb{C}[X_{SO(4,\mathbb{C})}(F_2)]$ *is a* $\mathcal{FT}_{SO(4,\mathbb{C})}(F_2)$*-module generated by* 1, $Q_4(\gamma_1,\gamma_2)$,

[2] A partial version of this result (with infinite generating sets) can be also found in [ATZ]

$Q_4(\gamma_1\gamma_2^{-1},\gamma_2)$, and $Q_4(\gamma_2\gamma_1^{-1},\gamma_1)$.
(3) None of the generators of part (2) can be expressed as a linear combination of others with coefficients in $\mathcal{FT}_{SO(4,\mathbb{C})}(F_2)$. Consequently, $\mathcal{FT}_{SO(4,\mathbb{C})}(F_2) \subsetneq \mathbb{C}[X_{SO(4,\mathbb{C})}(F_2)]$.

Consequently, for every group Γ of corank ≥ 2 (i.e. having F_2 as its quotient), $\mathcal{FT}_{SO(4,\mathbb{C})}(\Gamma)$ is a proper subalgebra of $\mathbb{C}[X_{SO(4,\mathbb{C})}(\Gamma)]$, cf. Corollary 15.

2. The relation between $X_G(\Gamma)$ and $X_{G/H}(\Gamma)$

Let $Hom^0(\Gamma,G)$ denote the connected component of the trivial representation in $Hom(\Gamma,G)$. Let $X_G^0(\Gamma)$ denote its image in $X_G(\Gamma)$.

Let H be a finite, central subgroup of G. Then $Hom(\Gamma,H)$ acts on $Hom(\Gamma,G)$ by multiplication,

$$\sigma \cdot \rho(\gamma) = \sigma(\gamma)\cdot\rho(\gamma),$$

for $\gamma \in \Gamma$. Let $Hom'(\Gamma,H)$ be the set of elements of $Hom(\Gamma,H)$ which map $Hom^0(\Gamma,G)$ to itself.

Proposition 1. *(1) The projection $G \to G/H$ induces an isomorphism $\phi : Hom^0(\Gamma,G)/Hom'(\Gamma,H) \to Hom^0(\Gamma,G/H)$.*
(2) Furthermore, ϕ descends to an isomorphism $\psi : X_G^0(\Gamma)/Hom'(\Gamma,H) \to X_{G/H}^0(\Gamma)$.

Proof. (1) Since the extension $\{e\} \to H \to G \to G/H \to \{e\}$ is central, it defines an element $\alpha \in H^2(G,H)$ such that $f : \Gamma \to G/H$ lifts to $\tilde{f} : \Gamma \to G$ if and only if $f^*(\alpha) = 0$ in $H^2(\Gamma,H)$, cf. [GM, Sec. 2]. Since $H^2(\Gamma,H)$ is discrete, the property of f being "liftable" is locally constant on $Hom(\Gamma,G)$ (in complex topology). Since the trivial representation is liftable, the above argument shows that $G \twoheadrightarrow G/H$ induces an epimorphism $Hom^0(\Gamma,G) \to Hom^0(\Gamma,G/H)$. That epimorphism factors to

$$\phi : Hom^0(\Gamma,G)/Hom'(\Gamma,H) \to Hom^0(\Gamma,G/H).$$

Since any two representations in $Hom^0(\Gamma,G)$ projecting to the same representation in $Hom^0(\Gamma,G/H)$ differ by an element of $Hom'(\Gamma,H)$, ϕ is 1-1.

(2) follows from the fact that the action of $Hom'(\Gamma,H)$ on $Hom^0(\Gamma,G)$ commutes with the G-action by conjugation. □

Proposition 2. *The projection $G \to G/H$ induces an isomorphism $\psi : X_G(F_N)/H^N \to X_{G/H}(F_N)$ for free groups, F_N.*

Proof. For G connected, $Hom(F_N,G) = G^N$ is connected and, hence, the statement is an immediate consequence of Proposition 1. Observe, however, that the only reason for considering a specific connected component of $Hom(\Gamma,G/H)$ was to make sure that representations $\Gamma \to G/H$ are liftable to G. Since all representations are liftable for Γ free, the proof above implies the statement of this proposition as well. □

3. Finite generating sets of $\mathbb{C}[X_{G/H}(\Gamma)]$

Let us assume that H is a cyclic group of order m of scalar matrices in a matrix group G. We will apply the above results to provide finite generator sets for the coordinate rings of G/H-character varieties. Since for every Γ the natural projection

$$\pi : F_N = \langle \gamma_1, ..., \gamma_N \rangle \to \Gamma$$

induces an epimorphism

$$\pi_* : \mathbb{C}[X_{G/H}(F_N)] \to \mathbb{C}[X_{G/H}(\Gamma)],$$

we will construct finite generating sets of $\mathbb{C}[X_{G/H}(\Gamma)]$ for free groups Γ only.

Given $\gamma \in F_N = \langle \gamma_1, ..., \gamma_N \rangle$, let $v(\gamma)$ be a vector in $(\mathbb{Z}/m)^N$, whose i-th component is the sum (mod m) of exponents of γ_i in γ.

Theorem 3. (1) *If $g_1, ..., g_k$ are elements of F_N such that $\sum_{i=1}^k v(g_i) = 0$, then there exists a unique $\lambda_{g_1,...,g_k} : X_{G/H}(F_N) \to \mathbb{C}$ such that $\tau_{g_1} \cdot ... \cdot \tau_{g_k} = \lambda_{g_1,...,g_k}\psi$, where $\psi : X_G(F_N) \to X_{G/H}(F_N)$ is the map of Proposition 2.*
(2) *Suppose that $\mathbb{C}[X_G(F_N)]$ is generated by τ_g, for g's in a set $B \subset F_N$. Let $\mathcal{M}$ be the set of all $\lambda_{g_1,...,g_k}$ for $g_1, ..., g_k \in B$ and $\sum_{i=1}^k v(g_k) = 0$. Then $\mathbb{C}[X_{G/H}(F_N)]$ is generated by $\mathcal{M}$.*

Proof of Theorem 3: (1) It is easy to see that for each $g_1, ..., g_k$ such that $\sum_{i=1}^k v(g_i) = 0$, $\tau_{g_1} \cdot ... \cdot \tau_{g_k}$ is H^N invariant. Hence, $\tau_{g_1} \cdot ... \cdot \tau_{g_k} \in \mathbb{C}[G^N]^{G \times H^N}$ and, by Proposition 2, it defines a function $\lambda_{g_1,...,g_k}$ in $\mathbb{C}[X_{G/H}(F_N)]$.

(2) Since τ_g generate $\mathbb{C}[G^N]^G$, for $g \in B$, every $f \in \mathbb{C}[X_{G/H}(F_N)] = \mathbb{C}[G^N]^{G \times H^N}$ can be written as

$$f = \sum_{i=1}^s f_i,$$

where each f_i is a monomial in τ_g, for $g \in B$. Let $s(f)$ be the smallest such s for given f.

We prove the statement of the theorem by contradiction. Suppose that there is an element of $\mathbb{C}[G^N]^{G \times H^N}$ which is not a polynomial expression in elements of $\mathcal{M}$. Choose such f with $s(f)$ as small as possible. Let $f = \sum_{i=1}^s f_i$ be a corresponding decomposition. A generator h of H acts on each f_i by a scalar multiplication. Let $h \cdot f_i = c_i \cdot f_i$. Then $c_s \neq 1$, since otherwise $f_s \in \mathcal{M}$ and $f - f_s$ is an element not generated by $\mathcal{M}$ with $s(f - f_s) < s = s(f)$. Since

$$f = h \cdot f = \sum_{i=1}^s c_i f_i,$$

$$c_s f - f = c_s f - \sum_{i=1}^s c_i f_i$$

and, hence,

$$f = \frac{1}{c_s - 1} \sum_{i=1}^{s(f)-1} (c_s - c_i) f_i,$$

contradicting the minimality of $s(f)$. $\square$

Since $\mathcal{M}$ contains $\lambda_{g_1,...,g_k}$ for an arbitrarily long sequences $g_1, ..., g_k$, the set $\mathcal{M}$ is usually infinite. We are going to reduce it to a finite generating set now.

Let $\mathcal{V}(m, N)$ be the set of all multisets[3] $\{v_1, v_2, ..., v_k\}$ of vectors in $(\mathbb{Z}/m)^N$, such that $\sum_{i=1}^k v_i = 0$, but no proper, non-empty submultiset of $\{v_1, ..., v_k\}$ adds up to 0. Let $\mathcal{M}' \subset \mathcal{M}$ be composed of functions $\lambda_{g_1,...,g_k}$ such that $g_1, ..., g_k \in B$ and $\{v(g_1), ..., v(g_k)\} \in \mathcal{V}(m, N)$. Since every element of $\mathcal{M}$ is a product of elements in $\mathcal{M}'$, we have

Corollary 4. $\mathbb{C}[X_{G/H}(F_N)]$ *is generated by the elements of* $\mathcal{M}'$.

$\mathcal{V}(m, N)$ is finite and, consequently, $\mathcal{M}'$ is finite as well. Indeed, we have:

Proposition 5. *The sequences in* $\mathcal{V}(m, N)$ *have length at most* m^N.

Proof. Suppose that $\{v_1, v_2, ..., v_k\} \in \mathcal{V}(m, N)$. Then $\sum_{i=1}^l v_i \neq 0$ in $(\mathbb{Z}/m)^N$, for all $l = 1, ..., k-1$. If $k > m^N$, then $\sum_{i=1}^{l_1} v_i = \sum_{i=1}^{l_2} v_i$ for some $l_1 < l_2 \leq k$ and $\sum_{i=l_1+1}^{l_2} v_i = 0$. $\square$

For example, $\mathcal{V}(2, 2)$ is composed of five multisets: $\{(0, 0)\}$, $\{(1, 0), (1, 0)\}$, $\{(0, 1), (0, 1)\}$, $\{(1, 1), (1, 1)\}$, $\{(1, 0), (0, 1), (1, 1)\}$.

The length of the longest sequence in $\mathcal{V}(m, N)$ is called the Davenport constant for the group $(\mathbb{Z}/m)^N$. No explicit formula for it is known. However, it is known that

$$N(m-1) + 1 \leq d(m, N) \leq (m-1)(1 + (N-1)m \ln m) + 1$$

and, furthermore, the equality on the left holds for m prime (and any N) and for $N = 2$ (and any m), [GLP].

Corollary 6. *Since* $\mathbb{C}[X_{SL(2,\mathbb{C})}(F_2)] = \mathbb{C}[\tau_{\gamma_1}, \tau_{\gamma_2}, \tau_{\gamma_1\gamma_2}]$, $X_{PSL(2,\mathbb{C})}(F_2)$ *is generated by* $g_1 = \tau_{\gamma_1}^2$, $g_2 = \tau_{\gamma_2}^2$, $g_3 = \tau_{\gamma_1\gamma_2}^2$ *and* $g_4 = \tau_{\gamma_1}\tau_{\gamma_2}\tau_{\gamma_1\gamma_2}$. *It is easy to see that* $g_1g_2g_3 - g_4^2$ *generates all other relations between these generators. Therefore,*

$$X_{PSL(2,\mathbb{C})}(F_2) = \mathbb{C}[g_1, g_2, g_3, g_4]/(g_1g_2g_3 - g_4^2).$$

$PSL(2, \mathbb{C})$-character varieties were studied in further detail in [HP].

Example 7. $SO(4, \mathbb{C}) = (SL(2, \mathbb{C}) \times SL(2, \mathbb{C}))/(\mathbb{Z}/2\mathbb{Z})$ *and* $SO(6, \mathbb{C}) = SL(4, \mathbb{C})/(\mathbb{Z}/2)$. *Therefore, the method of this section provides generating*

[3] A multiset is a "set" in which an element may appear more than once.

sets of coordinate rings of $SO(4,\mathbb{C})$- and $SO(6,\mathbb{C})$-character varieties, alternative to those obtained in [S2, Theorem 8]*. We will discuss $SO(4,\mathbb{C})$-character varieties in further detail in Sec. 5.*

Corollary 4 provide finite generating sets of coordinate rings of G/H-character varieties for $G = SL(n,\mathbb{C}), Sp(2n,\mathbb{C})$, and $SO(2n+1,\mathbb{C})$. (In the last two cases, H is either trivial or $H = \{\pm 1\}$.) The case of $G = SO(2n,\mathbb{C})$ is not much different. In this case, the coordinate ring $\mathbb{C}[X_G(\Gamma)]$ is generated by τ_g, for g in some $B \subset \Gamma$, and by $Q_{2n}(g_1,..,g_n)$, for $g_1,...,g_n$ in some $B' \subset \Gamma$, c.f. [S2, Thm. 8]. We say that τ_g has weight $v(g) \in (\mathbb{Z}/2)^N$ and that $Q_{2n}(g_1,..,g_n)$ has weight $v(g_1)+...+v(g_n)$.

The only non-trivial central subgroup of $SO(2n,\mathbb{C})$ is $H = \{\pm 1\}$ and for $PSO(2n,\mathbb{C}) = SO(2n,\mathbb{C})/H$ we have

Theorem 8. *$\mathbb{C}[X_{PSO(2n,\mathbb{C})}(F_N)]$ is generated by monomials in the above generators of $\mathbb{C}[X_{SO(2n,\mathbb{C})}(F_N)]$ of total weight 0 in $(\mathbb{Z}/2)^N$.*

The proof is a straightforward adaptation of that of Theorem 3. □

4. Trace algebras and full trace algebras for $G = SO(2n,\mathbb{C})$

The following theorem provides a useful description of the extension $\mathcal{T}_{SO(2n,\mathbb{C})}(\Gamma) \subset \mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)]$. Let M be any $2n \times 2n$ orthogonal matrix of determinant -1. Then conjugation by M determines a transformation σ on $X_{SO(2n,\mathbb{C})}(\Gamma)$. Since $M^2 \in SO(2n,\mathbb{C})$, σ is an involution. Furthermore, since every two matrices M (as above) are conjugated one with another by a matrix in $SO(2n,\mathbb{C})$, the involution σ does no depend on the choice of M.

Theorem 9. *(1) $\mathcal{T}_{SO(2n,\mathbb{C})}(\Gamma) = \mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)]^\sigma$ (the σ-invariant part of $\mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)]$).*
(2) $\sigma(Q_{2n}(g_1,...,g_n)) = -Q_{2n}(g_1,...,g_n)$, for any $g_1,...,g_n \in \Gamma$.
(3) For every epimorphism $\phi : \Gamma \to \mathbb{Z}$ and for every $\gamma \in \Gamma$ with $\phi(\gamma) \neq 0$, $Q_{2n}(\gamma,...,\gamma) \notin \mathcal{T}_{SO(2n,\mathbb{C})}(\Gamma)$.

Proof. (1) Consider the action of $\mathbb{Z}/2 = \{1,\sigma\}$ on $X_{SO(2n,\mathbb{C})}(\Gamma)$. The embedding $\eta : X_{SO(2n,\mathbb{C})}(\Gamma)/\mathbb{Z}/2 \to X_{O(2n,\mathbb{C})}(\Gamma)$ induces an epimorphism

$$\eta_* : \mathbb{C}[X_{O(2n,\mathbb{C})}(\Gamma)] \to \mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)]^\sigma.$$

By [FL, Thm A.1], [S2, Thm 5], $\mathbb{C}[X_{O(2n,\mathbb{C})}(\Gamma)]$ is generated by τ_γ's. Since η_* maps $\tau_\gamma \in \mathbb{C}[X_{O(2n,\mathbb{C})}(\Gamma)]$ to $\tau_\gamma \in \mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)]$, the statement follows.

(2) It suffices to show that for any $2n \times 2n$ matrices $X_1,...,X_n$

$$Q_{2n}(MX_1M^T,...,MX_nM^T) = -Q_{2n}(X_1,...,X_n).$$

Furthermore, by [S2, Prop. 10(2)], it is enough to assume that $X_1 = ... = X_n$. Hence, by [S2, Prop. 10(1)], this equality reduces to

$$Pf(M(X - X^T)M^T) = -Pf(X - X^T),$$

(Pf denotes the Pfaffian), which follows from the fact that

$$Pf(M(X - X^T)M^T) = Pf(X - X^T) \cdot Det(M).$$

(3) Let us consider the $SO(2n, \mathbb{C})$-character variety of $\mathbb{Z}$ first. By [S3],

$$\mathbb{C}[X_{SO(2n,\mathbb{C})}(\mathbb{Z})] = \mathbb{C}[x_1^{\pm 1}, ..., x_n^{\pm 1}, (x_1 \cdot ... \cdot x_n)^{\frac{1}{2}}]^W,$$

where W is the Weyl group of $SO(2n, \mathbb{C})$ acting on $x_1, ..., x_n$ by signed permutations with an even number of sign changes. By [S3, Lemma 7.4], $Q_{2n}(k, ..., k)$ corresponds to

$$i^n \cdot \sum_{\mu \in S_n} sign(\mu) \prod_{i=1}^{n} (x_{\mu(i)}^k - x_{\mu(i)}^{-k}).$$

For $k \neq 0$, this expression does not vanish in $\mathbb{C}[x_1^{\pm 1}, ..., x_n^{\pm 1}, (x_1 \cdot ... \cdot x_n)^{\frac{1}{2}}]^W$.

Since the induced projection $\phi_* : \mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)] \to \mathbb{C}[X_{SO(2n,\mathbb{C})}(\mathbb{Z})]$ maps $Q_{2n}(\gamma, ..., \gamma)$ to $Q_{2n}(k, ..., k) \neq 0$, the element $Q_{2n}(\gamma, ..., \gamma)$ is non-zero and, hence, by Theorem 9(2), it is not σ-invariant. Now the statement follows from Theorem 9(1). □

Corollary 10. *(1) For any $g_1, ..., g_n, g_1', ..., g_n' \in \Gamma$,*

$$Q_{2n}(g_1, ..., g_n) Q_{2n}(g_1', ..., g_n') \in \mathcal{T}_{SO(2n,\mathbb{C})}(\Gamma).$$

(2) $\mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)]$ is a $\mathcal{T}_{SO(2n,\mathbb{C})}(\Gamma)$-module generated by 1 and by $Q_{2n}(g_1, ..., g_n)$ for words $g_1, ..., g_n \in \Gamma$ in $\gamma_1, ..., \gamma_N$ of length at most $\nu_n - 1$ in which the number of inverses is not larger than the half of the length of the word.

Proof. (1) Since by Theorem 9(2), the product of any two Q's is σ-invariant, the statement follows from Theorem 9(1).
(2) follows from part (1) and from [S2, Thm 8]. □

Proposition 11. *For every group Γ, the full trace algebra $\mathcal{FT}_{SO(2n,\mathbb{C})}(\Gamma)$ is a $\mathcal{T}_{SO(2n,\mathbb{C})}(\Gamma)$-module generated by 1 and by $Q_{2n}(\gamma, ..., \gamma)$ for $\gamma \in \Gamma$.*

Proof. By [FH, Sec 23.2], the representation ring of $SO(2n, \mathbb{C})$ is generated by the exterior powers, $\wedge^k V$, of the defining ($2n$-dimensional) representation V and by the representations D_n^+, D_n^-, whose highest weights are twice that of the $\pm$-half-spin representations. (The last two representations are denoted by $D_\pm$ in [S2].)

For every $M \in M(2n, \mathbb{C})$ with eigenvalues $\lambda_1, ..., \lambda_{2n}$, the induced transformation on $\wedge^k \mathbb{C}^{2n}$ has trace $\sum_{i_1 < ... < i_k} \lambda_{i_1} \cdot ... \cdot \lambda_{i_k}$, which is a polynomial in $tr(M^d)$ for $d = 1, 2, 3, ...$ That polynomial depends on k only. Therefore, for every k, $\tau_{\gamma, \wedge^k V} \in \mathcal{T}_{SO(2n,\mathbb{C})}(\Gamma)$, implying that $\mathcal{FT}_{SO(2n,\mathbb{C})}(\Gamma)$ is a $\mathcal{T}_{SO(2n,\mathbb{C})}(\Gamma)$-algebra generated by $\tau_{\gamma, D_n^\pm}$, for $\gamma \in \Gamma$. Since $D_n^+ + D_n^-$ is a polynomial in V, cf. [FH, Sec 23.2], $\tau_{\gamma, D_n^+} + \tau_{\gamma, D_n^+} \in \mathcal{FT}_{SO(2n,\mathbb{C})}(\Gamma)$ and, consequently, $\mathcal{FT}_{SO(2n,\mathbb{C})}(\Gamma)$ is generated by expressions $\tau_{\gamma, D_n^+} - \tau_{\gamma, D_n^+}$. By [S2, Prop 10],

$$\tau_{\gamma, D_n^+} - \tau_{\gamma, D_n^+} = (2i)^{-n} (n!)^{-1} Q_{2n}(\gamma, ..., \gamma),$$

implying that $Q_{2n}(\gamma,...,\gamma)$, for $\gamma\in\Gamma$, are $\mathcal{T}_{SO(2n,\mathbb{C})}(\Gamma)$-algebra generators of $\mathcal{FT}_{SO(2n,\mathbb{C})}(\Gamma)$. Now the statement follows from Corollary 10(1). □

Theorem 12. *Let Γ be finite. If $Q_n(g_1,...,g_n)\neq 0$ for some $g_1,...,g_n\in\Gamma$ then $\mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)]$ is generated by 1 and by $Q_n(g_1,...,g_n)$ as an $\mathcal{T}_{SO(2n,\mathbb{C})}(\Gamma)$-module.*

Consequently, $\mathcal{FT}_{SO(2n,\mathbb{C})}(\Gamma)=\mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)]$ for every finite group Γ.

Proof of Theorem 12: Let $\mathcal{T}'_{SO(2n,\mathbb{C})}(\Gamma)$ be the $\mathcal{T}_{SO(2n,\mathbb{C})}(\Gamma)$-submodule of $\mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)]$ generated by 1 and $Q_n(g_1,...,g_n)$. It is a $\mathcal{T}_{SO(2n,\mathbb{C})}(\Gamma)$-algebra by Corollary 10(1).

Since for every $\rho:\Gamma\to SO(2n,\mathbb{C})$, $H^1(\Gamma, Ad\,\rho)=0$, $X_{SO(2n,\mathbb{C})}(\Gamma)$ is a discrete set, by [S1, Cor. 45(2)], and, hence, it is finite. Denote its cardinality by s.

Consequently, $\mathcal{T}'_{SO(2n,\mathbb{C})}(\Gamma)\simeq\mathbb{C}^t$, for some $t\leq s$. Now, the equality $\mathcal{T}'_{SO(2n,\mathbb{C})}(\Gamma)=\mathbb{C}^t=\mathbb{C}^s=\mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)]$, follows from the the lemma below. □

Lemma 13. *The elements of $\mathcal{T}'_{SO(2n,\mathbb{C})}(\Gamma)$ distinguish the points of $X_{SO(2n,\mathbb{C})}(\Gamma)$.*

Proof. Since the $O(2n,\mathbb{C})$-character varieties coincide with their trace algebras, any two points $[\rho_1]$, $[\rho_2]$ of $X_{SO(2n,\mathbb{C})}(\Gamma)$ which are not equal in $X_{O(2n,\mathbb{C})}(\Gamma)$ are distinguished by some τ_γ. If $[\rho_1]\neq[\rho_2]$ in $X_{SO(2n,\mathbb{C})}(\Gamma)$, but $[\rho_1]=[\rho_2]$ in $X_{SO(2n,\mathbb{C})}(\Gamma)$, then $[\rho_2]=\sigma[\rho_1]$, and, consequently, $[\rho_1],[\rho_2]$ are distinguished by $Q_n(g_1,...,g_n)$, by Theorem 9(2). □

Let us analyze the extensions $\mathcal{FT}_{SO(2n,\mathbb{C})}(\Gamma)\subset\mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)]$ in further detail now for $\Gamma=F_2$ and $n=2$:

Theorem 14. *(Proof in Sec. 6 and 7.)*
(1) $\mathbb{C}[X_{SO(4,\mathbb{C})}(F_2)]$ is a $\mathcal{T}_{SO(4,\mathbb{C})}(F_2)$-module generated by $1, Q_4(\gamma_i,\gamma_j)$, for $1\leq i\leq j\leq 2$, and by $Q_4(\gamma_1\gamma_2,\gamma_1\gamma_2)$, $Q_4(\gamma_1\gamma_2^{-1},\gamma_1\gamma_2^{-1})$, $Q_4(\gamma_1\gamma_2^{-1},\gamma_2)$, $Q_4(\gamma_2\gamma_1^{-1},\gamma_1)$.
(2) $\mathbb{C}[X_{SO(4,\mathbb{C})}(F_2)]$ is a $\mathcal{FT}_{SO(4,\mathbb{C})}(F_2)$-module generated by 1, $Q_4(\gamma_1,\gamma_2)$, $Q_4(\gamma_1\gamma_2^{-1},\gamma_2)$, and $Q_4(\gamma_2\gamma_1^{-1},\gamma_1)$.
(3) None of the generators of part (2) can be expressed as a linear combination of others with coefficients in $\mathcal{FT}_{SO(4,\mathbb{C})}(F_2)$. Consequently, $\mathcal{FT}_{SO(4,\mathbb{C})}(F_2)\subsetneq\mathbb{C}[X_{SO(4,\mathbb{C})}(F_2)]$.

Corollary 15. *For every group Γ of corank ≥ 2 (i.e. having F_2 as its quotient), $\mathcal{FT}_{SO(4,\mathbb{C})}(\Gamma)$ is a proper subalgebra of $\mathbb{C}[X_{SO(4,\mathbb{C})}(\Gamma)]$.*

Proof. The statement follows from the fact that every epimorphism $\phi:\Gamma\to F_2$ induces an epimorphism $\phi_*:\mathbb{C}[X_{SO(4,\mathbb{C})}(\Gamma)]\to\mathbb{C}[X_{SO(4,\mathbb{C})}(F_2)]$ mapping $\mathcal{FT}_{SO(4,\mathbb{C})}(\Gamma)$ to $\mathcal{FT}_{SO(4,\mathbb{C})}(F_2)$. □

Conjecture 16. *For every $n>2$ and every group Γ of corank ≥ 2, $\mathcal{FT}_{SO(2n,\mathbb{C})}(\Gamma)$ is a proper subalgebra of $\mathbb{C}[X_{SO(2n,\mathbb{C})}(\Gamma)]$.*

5. A PRESENTATION OF $\mathbb{C}[X_{SO(4,\mathbb{C})}(F_2)]$ IN TERMS OF GENERATORS AND RELATIONS

There is a natural isomorphism $SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) \to Spin(4,\mathbb{C})$ inducing the epimorphism

$$\phi : SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) \to SO(4,\mathbb{C}) \tag{1}$$

with the kernel $\{(I,I),(-I,-I)\}$, [GOV, Ch. 3.§2 Example 2]. This epimorphism can be defined as follows: Consider the action of $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ on $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$ by matrix multiplication. That action preserves the symmetric, non-degenerate product

$$((u_1,u_2),(v_1,v_2)) = det(u_1,v_1)det(u_2,v_2),$$

where $det(u,v)$ is the determinant of the 2×2 matrix composed of vectors u and v. Consequently, it induces the desired epimorphism ϕ.

Consider the following orthonormal basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$:

$$w_1 = \frac{1}{\sqrt{2}}(e_1 \otimes e_1 + e_2 \otimes e_2), \quad w_2 = \frac{i}{\sqrt{2}}(e_1 \otimes e_1 - e_2 \otimes e_2),$$

$$w_3 = \frac{i}{\sqrt{2}}(e_1 \otimes e_2 + e_2 \otimes e_1), \quad w_4 = \frac{1}{\sqrt{2}}(e_1 \otimes e_2 - e_2 \otimes e_1),$$

where $e_1 = (1,0)$, $e_2 = (0,1)$. A direct computation (which will be useful later) shows that for

$$((a_{ij}),(b_{ij})) \in SL(2,\mathbb{C}) \times SL(2,\mathbb{C}),$$

$\phi((a_{ij}),(b_{ij}))$ is given by the following matrix with respect to the above basis:

$$\phi((a_{ij}),(b_{ij})) = \tag{2}$$

$$\frac{1}{2} \cdot \begin{pmatrix} a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22} & ia_{11}b_{11} - ia_{12}b_{12} + ia_{21}b_{21} - ia_{22}b_{22} & ia_{12}b_{11} + ia_{11}b_{12} + ia_{22}b_{21} + ia_{21}b_{22} & -a_{12}b_{11} + a_{11}b_{12} - a_{22}b_{21} + a_{21}b_{22} \\ -ia_{11}b_{11} - ia_{12}b_{12} + ia_{21}b_{21} + ia_{22}b_{22} & a_{11}b_{11} - a_{12}b_{12} - a_{21}b_{21} + a_{22}b_{22} & a_{12}b_{11} + a_{11}b_{12} - a_{22}b_{21} - a_{21}b_{22} & ia_{12}b_{11} - ia_{11}b_{12} - ia_{22}b_{21} + ia_{21}b_{22} \\ -ia_{21}b_{11} - ia_{22}b_{12} - ia_{11}b_{21} - ia_{12}b_{22} & a_{21}b_{11} - a_{22}b_{12} + a_{11}b_{21} - a_{12}b_{22} & a_{22}b_{11} + a_{21}b_{12} + a_{12}b_{21} + a_{11}b_{22} & ia_{22}b_{11} - ia_{21}b_{12} + ia_{12}b_{21} - ia_{11}b_{22} \\ -a_{21}b_{11} - a_{22}b_{12} + a_{11}b_{21} + a_{12}b_{22} & -ia_{21}b_{11} + ia_{22}b_{12} + ia_{11}b_{21} - ia_{12}b_{22} & -ia_{22}b_{11} - ia_{21}b_{12} + ia_{12}b_{21} + ia_{11}b_{22} & a_{22}b_{11} - a_{21}b_{12} - a_{12}b_{21} + a_{11}b_{22} \end{pmatrix}.$$

We are going to use the above description of $SO(4,\mathbb{C})$ and the method of Sec. 2 to find a presentation of $\mathbb{C}[X_{SO(4,\mathbb{C})}(F_2)]$ in terms of generators and relations.

By Proposition 2,

$$X_{SO(4,\mathbb{C})}(F_2) = (X_{SL(2,\mathbb{C})}(F_2) \times X_{SL(2,\mathbb{C})}(F_2))/(\mathbb{Z}/2 \times \mathbb{Z}/2).$$

The above action of $\mathbb{Z}/2 \times \mathbb{Z}/2$ on $X_{SL(2,\mathbb{C})}(F_2) \times X_{SL(2,\mathbb{C})}(F_2)$ can be described as follows: $(\varepsilon_1,\varepsilon_2) \in \{\pm 1\} \times \{\pm 1\} = \mathbb{Z}/2 \times \mathbb{Z}/2$ sends the equivalence class of $\rho : \Gamma \to SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ to the equivalence class of ρ' such that

$$\rho'(\gamma_i) = (\varepsilon_i \rho_1'(\gamma_i), \varepsilon_i \rho_2'(\gamma_i))$$

for $i = 1, 2$.

Let us abbreviate the generators $\tau_{\gamma_1}, \tau_{\gamma_2}, \tau_{\gamma_1\gamma_2}$ of $\mathbb{C}[X_{SL(2,\mathbb{C})}(F_2)]$ by $\tau_1, \tau_2, \tau_{12}$. (Hence, $\mathbb{C}[X_{SL(2,\mathbb{C})}(F_2)] = \mathbb{C}[\tau_1, \tau_2, \tau_{12}]$.) We will denote the generators of

$$\mathbb{C}[X_{SL(2,\mathbb{C})}(F_2) \times X_{SL(2,\mathbb{C})}(F_2)] = \mathbb{C}[X_{SL(2,\mathbb{C})}(F_2)] \otimes \mathbb{C}[X_{SL(2,\mathbb{C})}(F_2)]$$

by $\tau_{1,i}, \tau_{2,i}, \tau_{12,i}$, where $i = 1, 2$ indicates the first or the second copy of $\mathbb{C}[X_{SL(2,\mathbb{C})}(F_2)]$.

Corollary 17.

$$\mathbb{C}[X_{SO(4,\mathbb{C})}(F_2)] = \mathbb{C}[X_{SL(2,\mathbb{C})}(F_2) \times X_{SL(2,\mathbb{C})}(F_2)]^{\mathbb{Z}/2\times\mathbb{Z}/2}$$

is generated by

$$a_{i,j,k} = \tau_{i,j}\tau_{i,k}, \quad b_{j,k} = \tau_{12,j}\tau_{12,k}, \quad \textit{for } i, j, k = 1, 2 \textit{ where } j \leq k,$$

and by

$$c_{i,j,k} = \tau_{1,i}\tau_{2,j}\tau_{12,k}, \quad \textit{for } i, j, k = 1, 2.$$

Proposition 18. $\mathbb{C}[X_{SO(4,\mathbb{C})}(F_2)]$ *is the quotient of the polynomial ring in the above* 17 *generators* $a_{i,j,k}, b_{i,j}, c_{i,j,k}$ *by the ideal generated by*

$$a_{i,j,k}a_{i,j',k'} - a_{i,j,j'}a_{i,k,k'}, \quad b_{j,k}b_{j',k'} - b_{j,j'}b_{k,k'}, \quad c_{i,j,k} \cdot c_{i',j',k'} - a_{1,i,i'}a_{2,j,j'}b_{k,k'},$$

$$a_{1,j,k}c_{i,j',k'} - a_{1,i,k}c_{j,j',k'}, \quad a_{2,j,k}c_{i,j',k'} - a_{2,j',k}c_{i,j,k'}, \quad b_{j,k}c_{i,j',k'} - b_{j,k'}c_{i,j',j},$$

where $a_{i,2,1} = a_{i,1,2}$ *and* $b_{2,1} = b_{1,2}$.

Proof. Since all $\tau_{i,j}$'s and $\tau_{12,i}$'s are algebraically independent, all relations between the generators of $\mathbb{C}[X_{SO(4,\mathbb{C})}(F_2)]$ follow from different decompositions of monomials in τ's into products of a's, b's, and c's. It is straightforward to check that any two such decompositions are related by the relations listed above. □

6. Proof of Theorem 14(1) and (2)

Consider the isomorphism $\phi : SL(2,\mathbb{C}) \times SL(2,\mathbb{C}))/\{\pm(I,I)\} \to SO(4,\mathbb{C})$ of Sec. 5.

Lemma 19. *If* $M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, *then the involution* σ *on* $SO(4,\mathbb{C})$, $\sigma(X) = M \cdot X \cdot M^{-1}$, *satisfies* $\sigma\phi(A,B) = \phi(B,A)$ *for any* $A, B \in SL(2,\mathbb{C})$.

Proof. Since

$$C_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } C_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

generate $SL(2,\mathbb{Z})$, which is a Zariski-dense subgroup of $SL(2,\mathbb{C})$, and since σ is an algebraic group automorphism, it is enough to verify the statement for $(A,B) = (C_i, C_j)$, $i, j = 1, 2$. A substitution of (C_i, C_j) into (2) yields

$$\phi(C_1, C_1) = \frac{1}{2}\begin{pmatrix} 3 & -i & 2i & 0 \\ -i & 1 & 2 & 0 \\ -2i & -2 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \ \phi(C_1, C_2) = \frac{1}{2}\begin{pmatrix} -1 & i & 0 & -2 \\ i & 1 & -2 & 0 \\ 0 & 2 & 1 & i \\ 2 & 0 & i & -1 \end{pmatrix},$$

$$\phi(C_2,C_1) = \frac{1}{2}\begin{pmatrix} -1 & i & 0 & 2 \\ i & 1 & -2 & 0 \\ 0 & 2 & 1 & -i \\ -2 & 0 & -i & -1 \end{pmatrix},\ \phi(C_2,C_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since

$$\sigma\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} & x_{13} & -x_{14} \\ x_{21} & x_{22} & x_{23} & -x_{24} \\ x_{31} & x_{32} & x_{33} & -x_{34} \\ -x_{41} & -x_{42} & -x_{43} & x_{44} \end{pmatrix},$$

we easily see that $\phi(C_1,C_1)$ and $\phi(C_2,C_2)$ are σ-invariant, while $\sigma(\phi(C_1,C_2)) = \phi(C_2,C_1)$. ◻

Corollary 20. *$\sigma(a_{i,j,k}) = a_{i,\hat{j},\hat{k}}$, $\sigma(b_{j,k}) = b_{\hat{j},\hat{k}}$, $\sigma(c_{i,j,k}) = c_{\hat{i},\hat{j},\hat{k}}$, where $\hat{1} = 2$ and $\hat{2} = 1$.*

Lemma 21. *If σ is an involution on a commutative $\mathbb{C}$-algebra R generated by $r_1,...,r_s$, then*
(1) R^σ is generated by $p_i = r_i + \sigma(r_i)$ and by q_iq_j, for $i,j = 1,...,s$, where $q_i = r_i - \sigma(r_i)$.
(2) As an R^σ-module, R is generated by 1 and by q_i for $i = 1,...,s$.

Proof. (1) Since p_i's and q_i's form a generating set of R, every element $r \in R^\sigma$ is of the form $r = \sum_k m_k$, where m_j's are monomials in p_i's and q_i's. Clearly $\sigma m = \pm m$ for every monomial m in that polynomial. By replacing r by $\frac{r+\sigma r}{2}$ if necessary, we can assume without loss of generality that all monomials, m_k, are σ-invariant. Therefore, they are products of p_i's and q_iq_j's.

(2) Clearly R is generated by q_i's as an R^σ-algebra. Since $q_iq_j \in R^\sigma$, the algebra R is generated by 1 and by q_i's as an R^σ-module. ◻

By Theorem 9(1), Corollary 20 and the above lemma, we have

Corollary 22. *(1) As an $\mathcal{T}_{SO(4,\mathbb{C})}(F_2)$-module, $\mathbb{C}[X_{SO(4,\mathbb{C})}(F_2)]$ is generated by*

$$t_0 = 1,\ t_1 = c_{1,1,2} - c_{2,2,1},\ t_2 = c_{1,2,1} - c_{2,1,2},\ t_3 = c_{1,2,2} - c_{2,1,1},$$

$$t_4 = a_{1,1,1} - a_{1,2,2},\ t_5 = a_{2,1,1} - a_{2,2,2},\ t_6 = b_{1,1} - b_{2,2},\ t_7 = c_{1,1,1} - c_{2,2,2}.$$

(This specific order of t_i's will prove convenient later.)
(2) As a $\mathbb{C}$-algebra, $\mathcal{T}_{SO(4,\mathbb{C})}(F_2)$ is generated by

$$a_{i,1,1} + a_{i,2,2},\ a_{i,1,2},\ \textit{for } i = 1,2,\ b_{1,2},\ b_{1,1} + b_{2,2},$$

$$c_{1,1,1} + c_{2,2,2},\ c_{1,1,2} + c_{2,2,1},\ c_{1,2,1} + c_{2,1,2},\ c_{1,2,2} + c_{2,1,1},$$

and by the products t_jt_k, for $j,k = 1,...,7$

Lemma 23. *$Q_4(g_1,g_2) = 4(\tau_{g_1,2}\tau_{g_2,2}\tau_{g_1g_2,1} - \tau_{g_1,1}\tau_{g_2,1}\tau_{g_1g_2,2})$ in $\mathbb{C}[X_{SO(4,\mathbb{C})}(F_2)]$.*

Proof. A computer algebra system computation based on the definition of $Q_4(\cdot,\cdot)$ and on formula (2). ◻

In what follows, we use two classical identities:

$$\tau_{g^{-1}} = \tau_g \text{ and } \tau_g\tau_h = \tau_{gh} + \tau_{gh^{-1}}$$

for $\tau_g, \tau_h \in \mathbb{C}[X_{SL(2,\mathbb{C})}(\Gamma)]$.

By the lemma above,

$$Q_4(g,g) = 4(\tau_{g,2}^2\tau_{g^2,1} - \tau_{g,1}^2\tau_{g^2,2}) = 4(\tau_{g,2}^2(\tau_{g,1}^2-2) - \tau_{g,1}^2(\tau_{g,2}^2-2)) = 8(\tau_{g,1}^2 - \tau_{g,2}^2).$$

Consequently,

(3)
$$t_1 = -\frac{1}{4}Q_4(\gamma_1,\gamma_2),\ t_4 = \frac{1}{8}Q_4(\gamma_1,\gamma_1),\ t_5 = \frac{1}{8}Q_4(\gamma_2,\gamma_2),\ t_6 = \frac{1}{8}Q_4(\gamma_1\gamma_2,\gamma_1\gamma_2).$$

Since

$$\frac{1}{4}Q_4(\gamma_2\gamma_1^{-1},\gamma_1) = \tau_{\gamma_2\gamma_1^{-1},2}\tau_{1,2}\tau_{2,1} - \tau_{\gamma_2\gamma_1^{-1},1}\tau_{1,1}\tau_{2,2} =$$
$$(\tau_{1,2}\tau_{2,2} - \tau_{12,2})\tau_{1,2}\tau_{2,1} - (\tau_{1,1}\tau_{2,1} - \tau_{12,1})\tau_{1,1}\tau_{2,2} =$$
$$\tau_{2,1}\tau_{2,2}(\tau_{1,2}^2 - \tau_{1,1}^2) - (\tau_{1,2}\tau_{2,1}\tau_{12,2} - \tau_{1,1}\tau_{2,2}\tau_{12,1}) = -a_{2,1,2}t_4 + t_2,$$

we have

$$t_2 = \frac{1}{4}Q_4(\gamma_2\gamma_1^{-1},\gamma_1) + a_{2,1,2}t_4. \tag{4}$$

Analogously,

$$t_3 = -\frac{1}{4}Q_4(\gamma_1\gamma_2^{-1},\gamma_2) - a_{1,1,2}t_5 \tag{5}$$

and, finally, similar computations show that

$$t_7 = \frac{1}{4}\left(t_4(a_{2,1,1}+a_{2,2,2}) + t_5(a_{1,1,1}+a_{1,2,2})\right) + \frac{1}{2}t_6 - \frac{1}{16}Q_4(\gamma_1\gamma_2^{-1},\gamma_1\gamma_2^{-1}).$$

Corollary 22(1) together with these identities implies part (1) of Theorem 14. Now, part (2) follows immediately from part (1) and from Proposition 11. □

7. Proof of Theorem 14(3)

We proceed the proof with several preliminaries.

Lemma 24. *The following identities hold in $\mathbb{C}[X_{SO(4,\mathbb{C})}(F_2)]$ for every $\gamma \in F_2$:*
(1) $\tau_\gamma = \tau_{\gamma,1}\cdot\tau_{\gamma,2}$. (As before, τ_γ is an abbreviation for $\tau_{\gamma,\mathbb{C}^4}$, where $\mathbb{C}^4$ is the defining representation of $SO(4,\mathbb{C})$.)
(2) One of $\tau_{\gamma,D_2^+}, \tau_{\gamma,D_2^-}$ equals to $\tau_{\gamma,1}^2 - 1$ and the other to $\tau_{\gamma,2}^2 - 1$.

Proof. (1) Let $\pi_i : SL(2,\mathbb{C})\times SL(2,\mathbb{C}) \to SL(2,\mathbb{C})$ by the projection on the i-th factor, for $i = 1,2$. Then $\pi_1\otimes\pi_2$ is an irreducible 4-dimensional representation of $SL(2,\mathbb{C})\times SL(2,\mathbb{C})$. Since it sends $(-I,-I)$ to the identity, it factors to an irreducible 4-dimensional representation of $SO(4,\mathbb{C})$. Since the defining representation is the only irreducible 4-dimensional representation

of $SO(4,\mathbb{C})$ (up to conjugation), $Tr\pi_1 \otimes \pi_2(A_1, A_2) = Tr(\phi(A_1, A_2))$ for every $A_1, A_2 \in SL(2,\mathbb{C})$, where ϕ is the representation (1). Since

$$Tr\pi_1 \otimes \pi_2(A_1, A_2) = TrA_1 \cdot TrA_2,$$

the statement follows.

(2) Let η be an irreducible 3-dimensional representation of $SL(2,\mathbb{C})$. (η is unique up to conjugation.) It is easy to see that $\eta\pi_1$ and $\eta\pi_2$ factor to two nonequivalent irreducible representations of

$$SO(4,\mathbb{C}) = (SL(2,\mathbb{C}) \times SL(2,\mathbb{C}))/\pm(I, I).$$

By the classification of representations of $SO(4,\mathbb{C})$ (as described in the proof of Proposition 11) the only irreducible 3-dimensional representations of $SO(4,\mathbb{C})$ are D_2^+ and D_2^-. Since $Tr\eta(A) = (TrA)^2 - 1$, the statement follows. □

Proposition 25. *(1)* $\tau_{i,j}^2, \tau_{12,i}^2, \tau_{i,1}\tau_{i,2}, \tau_{12,1}\tau_{12,2}$, $\tau_{1,i}\tau_{2,i}\tau_{12,i}$, *for* $i, j \in \{1, 2\}$,

$$\tau_{1,1}\tau_{2,2}\tau_{12,1} + \tau_{1,2}\tau_{2,1}\tau_{12,2}, \quad \tau_{1,1}\tau_{2,2}\tau_{12,2} + \tau_{1,2}\tau_{2,1}\tau_{12,1},$$

and

$$\tau_{1,1}\tau_{2,1}\tau_{12,2} + \tau_{1,2}\tau_{2,2}\tau_{12,1}$$

belong to $\mathcal{FT}_{SO(4,\mathbb{C})}(F_2)$.
(2) $\mathcal{FT}_{SO(4,\mathbb{C})}(F_2)$ *is generated by the above elements as a* $\mathbb{C}$*-algebra.*

Proof of Proposition 25(1): $\tau_{i,j}^2, \tau_{12,i}^2, \tau_{i,1}\tau_{i,2}, \tau_{12,1}\tau_{12,2} \in \mathcal{FT}_{SO(4,\mathbb{C})}(F_2)$ by Lemma 24.

By squaring both sides of

$$\tau_{\gamma_1^2\gamma_2} = \tau_1\tau_{12} - \tau_2$$

we obtain

$$\tau_1\tau_2\tau_{12} = -\frac{1}{2}\left(\tau_{\gamma_1^2\gamma_2}^2 - \tau_1^2\tau_{12}^2 - \tau_2^2\right)$$

in $\mathbb{C}[X_{SL(2,\mathbb{C})}(F_2)]$. Therefore, $\tau_{1,i}\tau_{2,i}\tau_{12,i} \in \mathcal{FT}_{SO(4,\mathbb{C})}(F_2)$, for $i = 1, 2$, by the Lemma 24(2). Similarly, by applying $\gamma = \gamma_1^2\gamma_2$ to Lemma 24(1), we see that

$$\tau_{1,1}\tau_{2,2}\tau_{12,1} + \tau_{1,2}\tau_{2,1}\tau_{12,2} = \tau_{1,1}\tau_{1,2}\tau_{12,1}\tau_{12,2} + \tau_{2,1}\tau_{2,2} - \tau_{\gamma_1^2\gamma_2,1}\tau_{\gamma_1^2\gamma_2,2}$$

belongs to $\mathcal{FT}_{SO(4,\mathbb{C})}(F_2)$ as well. In the same vein, we prove that

$$\tau_{1,1}\tau_{2,2}\tau_{12,2} + \tau_{1,2}\tau_{2,1}\tau_{12,1},\ \tau_{1,1}\tau_{2,1}\tau_{12,2} + \tau_{1,2}\tau_{2,2}\tau_{12,1} \in \mathcal{FT}_{SO(4,\mathbb{C})}(F_2)$$

by taking $\gamma = \gamma_1\gamma_2^2$ and $\gamma = \gamma_1^{-1}\gamma_2$, respectively. □

In order to prove part (2) of Proposition 25 consider a $\mathbb{Z}/2 \times \mathbb{Z}/2$ action on $X_{SL(2,\mathbb{C})}(F_2)$ defined for $(k_1, k_2) \in \mathbb{Z}/2 \times \mathbb{Z}/2$ by $(k_1, k_2) \cdot [\rho] = [\rho']$, where $\rho'(\gamma_i) = (-1)^{k_i}\rho(\gamma_i)$, for $i = 1, 2$. That action defines a $\mathbb{Z}/2 \times \mathbb{Z}/2$-grading on $R = \mathbb{C}[X_{SL(2,\mathbb{C})}(F_2)]$, whose homogeneous components we denote by $R_{i,j}$, $i, j \in \mathbb{Z}/2$. More concretely, the elements $f \in R_{i,j}$, for $i, j \in \{0, 1\}$, have the

property that $(1,0)\cdot f=(-1)^i f$ and $(0,1)\cdot f=(-1)^j f$ for $i,j\in\mathbb{Z}/2$. In particular, $deg(\tau_1)=(1,0)$, $deg(\tau_2)=(0,1)$ and $deg(\tau_{12})=(1,1)$.

The following is straightforward:

Lemma 26. *$R_{0,0}=\mathbb{C}[\tau_1^2,\tau_2^2,\tau_{12}^2,\tau_1\tau_2\tau_{12}]$ and, as $R_{0,0}$-modules,*
$R_{1,0}$ is generated by τ_1 and $\tau_2\tau_{12}$,
$R_{0,1}$ is generated by τ_2 and $\tau_1\tau_{12}$, and
$R_{1,1}$ is generated by τ_{12} and $\tau_1\tau_2$.

Let $R_{0,0,i}$ be the subalgebra of $\mathbb{C}[X_{SO(4,\mathbb{C})}(F_2)]$ generated by the generators of $R_{0,0}$ with the second index i: $\tau_{1,i}^2,\tau_{2,i}^2,\tau_{12,i}^2,\tau_{1,i}\tau_{2,i}\tau_{12,i}$ for $i=1,2$. Similarly, let $R_{1,0,k}$ be an $R_{0,0,k}$-module generated by $\tau_{1,k}$ and $\tau_{2,k}\tau_{12,k}$, let $R_{0,1,k}$ be an $R_{0,0,k}$-module generated by $\tau_{2,k}$ and $\tau_{1,k}\tau_{12,k}$, and let $R_{1,1,k}$ be an $R_{0,0,k}$-module generated by $\tau_{12,k}$ and $\tau_{1,k}\tau_{2,k}$.

Proof of Proposition 25(2): We need to prove that $\mathcal{FT}_{SO(4,\mathbb{C})}(F_2)\subset\mathcal{S}$, where $\mathcal{S}$ denotes the $\mathbb{C}$-subalgebra of $\mathbb{C}[X_{SO(4,\mathbb{C})}(F_2)]$ generated by the elements listed in part (1).

Proof that $\tau_{\gamma,D_2^\pm}\in\mathcal{S}$ for every $\gamma\in F_2$: By Lemma 24(1), $\tau_{\gamma,D_2^\pm}=\tau_{\gamma,i}^2-1$ for $i=1$ or 2. Since τ_γ^2 is invariant under the $\mathbb{Z}/2\times\mathbb{Z}/2$ action, $\tau_\gamma^2\in R_{0,0}$ and $\tau_{\gamma,D_2^\pm}\in R_{0,0,i}$. Since $R_{0,0,1}$ and $R_{0,0,2}$ are subalgebras of $\mathcal{S}$, the statement follows.

Proof that $\tau_\gamma\in\mathcal{S}$: Consider the epimorphism $\eta: F_2\to\mathbb{Z}/2\times\mathbb{Z}/2$ sending γ_1 to $(1,0)$ and γ_2 to $(0,1)$. Since $(1,0)\cdot\tau_\gamma\in\mathbb{C}[X_{SL(2,\mathbb{C})}(F_2)]$ is either τ_γ or $-\tau_\gamma$ depending on the first component of $\eta(\gamma)$ and $(0,1)\cdot\tau_\gamma$ is either τ_γ or $-\tau_\gamma$ depending on the second component of $\eta(\gamma)$, we see that $\tau_\gamma\in R_{\eta(\gamma)}$ for every $\gamma\in F_2$.

For every γ, consider a three-variable polynomial p such that $\tau_\gamma=p(\tau_1,\tau_2,\tau_{12})$. By Lemma 24(1), $\tau_{\gamma,\mathbb{C}^4}=\tau_{\gamma,1}\cdot\tau_{\gamma,2}$. The following lemma completes the statement of Proposition 25(2). □

Lemma 27. *For every p as above, $p(\tau_{1,1},\tau_{2,1},\tau_{12,1})\cdot p(\tau_{1,2},\tau_{2,2},\tau_{12,2})\in\mathcal{S}$.*

Proof. We consider the four possible values of $\eta(\gamma)$. If $\eta(\gamma)=(0,0)$, then the statement is obvious since $R_{0,0,1},R_{0,0,2}\subset\mathcal{S}$. If $\eta(\gamma)=(1,0)$, then $p(\tau_{1,1},\tau_{2,1}\tau_{12,1})\cdot p(\tau_{1,2},\tau_{2,2}\tau_{12,2})$ belongs to an S-module generated by

$$\tau_{1,1}\tau_{1,2},\ \tau_{2,1}\tau_{12,1}\tau_{2,2}\tau_{12,2} \text{ and } \tau_{1,1}\tau_{2,2}\tau_{12,2}+\tau_{1,2}\tau_{2,1}\tau_{12,1}.$$

All these elements belong to $\mathcal{S}$. One can perform a similar verification for $\eta(\gamma)=(0,1)$ and for $(1,1)$. □

Proof of Theorem 14(3): Define a $\mathbb{Z}_{\geq 0}$-grading on $\mathbb{C}[\tau_{i,j},\tau_{12,j},\ i,j=1,2]$ by declaring that all $\tau_{i,j}$'s and $\tau_{12,j}$'s have degree 1. Note that $\mathbb{C}[X_{SO(4,\mathbb{C})}(F_2)]$ is a graded subalgebra of $\mathbb{C}[\tau_{i,j},\tau_{12,j},\ i,j=1,2]$. Since the generators of $\mathcal{FT}_{SO(4,\mathbb{C})}(F_2)$ of Proposition 25 are homogeneous, $\mathcal{FT}_{SO(4,\mathbb{C})}(F_2)$ is in turn a graded subalgebra of $\mathbb{C}[X_{SO(4,\mathbb{C})}(F_2)]$.

Suppose $Q_4(\gamma_2\gamma_1^{-1}, \gamma_1)$ is a $\mathcal{FT}_{SO(4,\mathbb{C})}(F_2)$-linear combination of 1, $Q_4(\gamma_1, \gamma_2)$, $Q_4(\gamma_1\gamma_2^{-1}, \gamma_2)$. Then from equalities (3), (4), (5), we see that t_3 is a $\mathcal{FT}_{SO(4,\mathbb{C})}(F_2)$-linear combination of t_0, t_1, t_2. Since $deg(t_0) = 0, deg(t_1) = deg(t_2) = deg(t_3) = 3$ and $\mathcal{FT}_{SO(4,\mathbb{C})}(F_2)$ is graded, $t_3 = c_0t_0 + c_1t_1 + c_2t_2$, for some $c_0, c_1, c_2 \in \mathcal{FT}_{SO(4,\mathbb{C})}(F_2)$ such that $deg(c_0) = 3$ and $deg(c_1) = deg(c_2) = 0$. Therefore, a non-trivial $\mathbb{C}$-linear combination of t_1, t_2, t_3 belongs to $\mathcal{FT}_{SO(4,\mathbb{C})}(F_2)$. Since such linear combination is a homogeneous element of degree 3, it has to coincide with a $\mathbb{C}$-linear combination of the generators of $\mathbb{C}[X_{SO(4,\mathbb{C})}(F_2)]$ of degree 3 listed in Proposition 25. It is easy to see that it is impossible, since $\tau_{i,j}$'s and $\tau_{12,j}$'s are algebraically independent.

The proof of the non-redundancy of the other three generators of Theorem 14(2) is analogous. □

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